Section 3.5: Limits at Infinity

When analyzing a graph, it is often important to consider what happens as *x* becomes very large in the positive or negative directions. A function may grow without bound, may oscillate between a set of values, or it may approach a certain value (i.e. have a horizontal asymptote).

If a function becomes arbitrarily close to a certain value *L* as *x* approaches ±∞, we say that the function has a limit at infinity of *L*:

$$\lim_{x \to \pm\infty} f(x) = L$$

Rational polynomial functions

Suppose that *R*(*x*) is a rational polynomial function of the form

$$R(x) = \frac{a x^n + b x^{n-1} + c x^{n-2} \ldots}{A x^m + B x^{m-1} + C x^{m-2} \ldots}$$

where *n* and *m* are positive integers and *a*, *b*, *c*... and *A*, *B*, *C* … are constants. As *x* approaches very large numbers, all numbers in the numerator and denominator can be ignored except for those with the highest power of *x*. Therefore, if the numerator is of higher degree than than the denominator, the function will approach ±∞, and in the reverse case, the function will approach zero. If the numerator and denominator are of the same degree, then the limit will be the ratio of the leading coefficients. This can be written symbolically as

$$\lim_{x \to \pm\infty} R(x) = \begin{cases} \pm\infty & \text{if } n > m \\ \frac{a}{A} & \text{if } n = m \\ 0 & \text{if } n' <' m \end{cases}$$

It is possible to apply a similar analysis to many other functions. For example, it is possible to rewrite

$$f(x) = \frac{x}{\sqrt{x^2 + 2x + 1}} \quad \to \quad \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$$

as *x* approaches very large numbers. In this case, notice that the infinite limits in the positive and negative directions are unequal. Namely,

$$\lim_{x \to \infty} f(x) = 1 \text{ , but } \lim_{x \to -\infty} f(x) = -1$$

Many other functions can be analyzed by rewriting as a rational function. For example, plugging *x* = ∞ into

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 + 3} \right)$$

results an ambiguous ∞ - ∞. This limit may be evaluated, however, by rewriting as a fraction with a denominator of 1 and multiplying by the conjugate:

$$\lim_{x\to\infty} \frac{x - \sqrt{x^2 - 3}}{1} \cdot \frac{x + \sqrt{x^2 - 3}}{x + \sqrt{x^2 - 3}} = \lim_{x\to\infty} \frac{3}{x + \sqrt{x^2 - 3}} = 0$$

Other infinite limits can be evaluated by rewriting an infinite as a non-infinite limit by making a substitution of variables (see exercises 39 and 40).

End behavior

While limits at infinity are useful in analyzing a graph, there are times when it is necessary to be more specific. Is a function approaching infinity at a constant rate, or at a quadratic rate? Is the function approaching a horizontal asymptote from above or from below?

Consider the function

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}$$

It can be shown that $f(x)$ approaches the line $y = x$ for large values of x, by performing long division to rewrite $f(x)$ as $x + \frac{4}{x-2}$. As x becomes very large, the second term goes to zero and, therefore $f(x)$ approaches x. We would say that $y = x$ is a *slant asymptote* of this function.

Moreover, the function approaches this asymptote from above, as can be shown by examining the second derivative

$$f''(x) = \frac{8}{(x-2)^2}$$

which is positive for all $x > 2$.

Section 3.6: A Summary of Curve Sketching

Analysis of extrema, concavity and end behavior provide a number of useful tools for sketching graphs of functions

Here is a (non-exhaustive) of properties that might be considered while graphing a function

1.	x- and y- intercepts	Section P.1
2.	symmetry	Section P.1
3.	domain and range	Section P.3
4.	continuity	Section 1.4
5.	vertical asymptotes	Section 1.5
6.	differentiability	Section 2.1
7.	critical points and relative extrema	Section 3.1
8.	concavity and points of inflection	Section 3.4
9.	end behavior and horizontal asymptotes	Section 3.5

For a graph to be considered complete, it should include all of the above features. For other functions there may be other properties of interest (e.g. period, amplitude and the like for functions involving trigonometric terms).

When drawing a graph, it is considered good practice to make a list of such properties before beginning the actual sketch. This will typically include an initial calculation calculating the first and second and calculating critical numbers and possible points of inflection, and application of tests to classify these points appropriately. However, it will not be necessary to perform all of these calculations for every problem, so it is advisable to use algebraic reasoning to determine which numbers it is worthwhile to calculate and which not.

Section 3.7: Optimization problems:

One of the most commonly used applications of calculus is the optimization of certain quantities. This may include things like maximizing strength or efficiency, minimizing time or volume, maximizing profit, minimizing cost etc.

The procedure for solving an optimization problem is as follows:

1. Identify the variable to be maximized and the quantities on which this variable depends.
2. Write an equation for the variable to be optimized—this is called the *primary equation.* (Drawing a sketch is often helpful for this step.)
3. If necessary, use constraint equations to rewrite primary equation in terms of a single independent variable.
4. Determine the practical domain of the function (i.e. the range of values for which the stated problems makes sense). This will give you the interval over which to test your function.
5. Determine the desired maxima or minima over the interval using the techniques of Sections 3.1 through 3.4.

Section 3.8: Newton's Method

In pre-calculus, one would typically finds the zeros of a function simply by setting the function equal to zero and solving for *x*. However, in many cases, such a procedure is difficult or impossible to perform. When this happens, it may be necessary to perform numerical approximations.

One such technique is known as *Newton's Method*. In this procedure, one makes an initial estimate x_1 of the root (sketching a graph may be helpful for this purpose) and generates a tangent line to the graph at that point. Assuming that the root of the tangent line is close to the root of the actual function, one may then use the root of the tangent line as an improved estimate (see Figure 1).

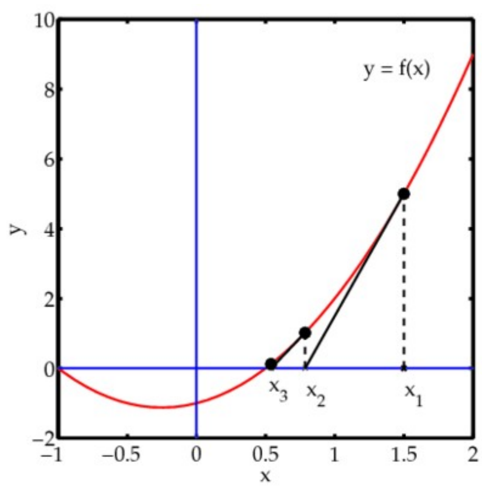

Figure 1: Two iterations of Newton's Method, showing an initial guess x_1 and two subsequent approximations.

If the function whose root is to be calculated is $f(x)$, and the initial estimate is x_1, then the equation of the tangent line is

$$y - f(x_i) = f'(x_i)(x - x_i)$$

Setting $y = 0$ and solving for x yields

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

There is no limit to the number of times this process may be repeated; the more iterations of Newton's Method you use, the more accurate your answer will become. More generally, if the current estimate is x_n, then the subsequent estimate x_{n+1} will be given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As these calculations tend to quickly become tedious, it is convenient to use a calculator program to perform the repeated calculations automatically.

Failure to converge

Newton's Method is not always successful at finding a root. If many iterations of Newton's Method fail to converge to a fixed estimate, it is possible that the initial estimate was not close enough to the actual root. Try again with a different initial guess.

Other functions (such as $y = x^{1/3}$) may fail to converge because they are too steep in the vicinity of the root. Technically, it can be proven that Newton's method will succeed only if

$$\left| \frac{f(x) f''(x)}{[f'(x)]^2} \right| < 1$$

on some open interval containing the root.

Section 3.9: Differentials

A *differential* is a defined as an infinitesimally small change in a number and is denoted *dx*.

To find an equation relating differentials to each other, we start with the slope equation

$$m = \frac{\Delta y}{\Delta x}$$

which we rearrange to

$$\Delta y = m \Delta x$$

As the change in x approaches zero, Δx becomes the differential dx. Meanwhile, if y is a continuous function of x, then Δx approaches a small change in that function df. Moreover, as Δx and Δy both approach zero, the average slope m approaches the derivative $f'(x)$. Therefore, we have

$$\boxed{dy = f'(x)\,dx}$$

Tangent line approximation

If Δx and Δy are relatively small then the average slope $\Delta y/\Delta x$ is approximately equal to the derivative $f'(x)$. Therefore, the expression

$$\Delta y = m \Delta x$$

can be replaced with

$$\Delta y \approx f'(a) \Delta x$$

Where a is the point at which the slope was calculated. This means that the change in the y-value a function is approximately equal to the slope multiplied by the change in x, as long as a is not too far from the point at which the function is to be approximated (see F**igure 2**).

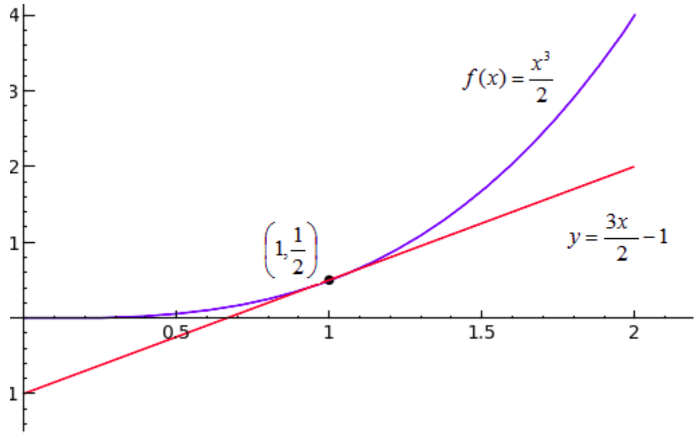

Figure 2: The tangent line approximation to the function $f(x) = x^3/2$. at $x = 1$. As long as the x value is not very far from 1, the tangent line (red) gives a good approximation of the function (blue).

Replacing Δx in the previous expression with $x - a$ and Δy with $f(x) - f(a)$, we can see that equation of the approximating function

$$f(x) \approx f(a) + f'(a)(x - a)$$

which is the equation of the tangent line to the curve at a.

Error analysis

Differentials may be interpreted as uncertainty or "errors" in measurements. If two variables are related by a function $A = f(B)$, then the uncertainty in A, ΔA is related to the uncertainty in B by

$$\Delta A \approx \frac{dA}{dB} \Delta B$$

Properties of differentials

Differentials possess many properties in common with derivatives:

Constant multiple	$d[cu] = c\,du$
Sum or difference	$d[u \pm v] = du \pm dv$
Product	$d[uv] = u\,dv + v\,du$
Quotient	$d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$